The Quandle and Group for General Pairs of Spaces*

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Abstract

Joyce has shown that the fundamental quandle of a classical knot can be derived from consideration of the fundamental group and the peripheral structure of the knot, and also that the group and much of the peripheral structure can be recovered from the quandle. We generalize these results to arbitrary dimensions, and also to virtual and welded knots and arcs.

1 Quandles

A *quandle* (sometimes called a *distributive groupoid*)[5, 7] is a non-associative algebraic structure which is particularly connected to the study of classical knots. In particular a quandle may be defined as a set Q together with an left-invertible binary operation which we will write by exponentiation a^b obeying the following relations:

$$a^a = a \tag{1}$$

$$(a^b)^c = (a^b)^{b^c} \tag{2}$$

Note that left invertibility is equivalent to also requiring that the equation $x^a = b$ with variable x should have exactly one solution. These relations are analogues of the Reidemeister moves for classical knots. In particular, Eq. 1 reflects the first Reidemeister move, left invertibility reflects the second, and Eq. 2 reflects the third. For this reason, they are particularly suited to the study of knots. Joyce[5] showed in particular that there is a quandle naturally associated to any knot in S^3 which is a nearly complete (up to mirror-reversal) invariant of the knot. The

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quandle associated to a classical knot is a special case of Joyce's quandle associated to any codimension two knotting. We briefly review this construction here, referring the reader to Joyce's exposition for further details.

Let M be a path-connected manifold, and let K be a codimension two knotting (neither M nor K need to be spheres). Let N be a tubular neighborhood of K. Then $M-N$ is a manifold with boundary. One component of ∂M will be $\partial\overline{N}$. Let P be the subgroup of $\pi_1(M-N)$ which is the image of the homomorphism induced by the injection $\partial\overline{N} \to M-N$. We refer to this as the *peripheral* subgroup. Note that P is defined only up to conjugation; any of its conjugates are also peripheral. Now suppose that in P there is a preferred element m, the *meridian*. In particular for any codimension two knot, the meridian is canonically defined up to conjugation. We may now define the quandle Q associated to the pair (M, N) and the chosen meridian. The elements of Q are homotopy classes of paths that start at the basepoint of M (that is, the basepoint chosen for the fundamental group) and end in $\partial\overline{N}$, where homotopies are required to preserve these two conditions. The quandle operation on two elements of $Q(M-N)$ is defined geometrically as illustrated in Fig. 1. In particular, $[q]^{[q']}$ is the homotopy class of paths containing the path $q'\overline{q}mq$, where the overbar indicates that the path is followed in reverse. It is a striaghtforward exercise to show that this is well-defined and meets the definition of a quandle operation. Note that there is an action $\pi_1(M-N)$ action

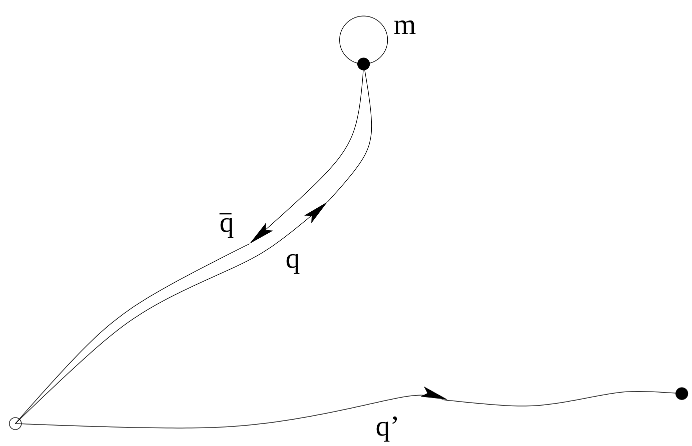

Figure 1: The quandle operation for the fundamental quandle of a pair of spaces. The open circle indicates the basepoint of M, while the black circles represent points on $\partial\overline{N}$.

on $Q(M-N)$: $[q]g$ is the equivalence class including the homotopy class of the path qg. We follow the notation of Joyce for this action.

2 Quandles From Groups

Given a group G, a subgroup P, and an element m in the center $Z(P)$, Joyce defines a quandle as follows. The underlying set is the set of right cosets of P, $P\backslash G$. We define the quandle operation such that $Pg^{Ph} = P(gh^{-1}mh)$. This is well-defined, because if $h' \in Ph$, then $h' = ah$ for some $a \in P$. Then

$$h'^{-1}mh' = h^{-1}a^{-1}mah = h^{-1}mh, \tag{3}$$

where the last equality follows from $m \in Z(P)$. We will denote a quandle constructed in this way by $(P\backslash G, m)$, or simply $P\backslash G$ when there is a canonical choice for m (as there is when m is a meridian and P a peripheral subgroup).

3 Group Actions on Quandles

The work in this section is largely a generalization of the work of Joyce[5] for the case of classical knots.

Consider a codimension two pair (M, K), and the tubular neighborhood N of K. Then we can define the quandle $Q = Q(M, K)$ geometrically as in the first section. On the other hand we can define the quandle G to be the quandle $(P\backslash\pi_1(M - K), m)$ for P some arbitrarily chosen peripheral subgroup of $\pi_1(M - K)$, and m the meridian in P.

Now π_1 acts on Q by setting $[q]g$ to be the equivalence class including the homotopy class of the path qg. m is an element of the fundamental group, but we may also define an analogous element m_Q in the fundamental quandle to be the quandle element such that $\overline{q}mq$ is the meridian of P.

Lemma 1 *Every element $q \in Q$ may be obtained as $m_Q g$ for some element $g \in \pi_1(M - N)$.*

Proof: We choose representatives of q, m_Q whose endpoints on ∂N agree. Then the path $m_Q\overline{m_Q}q$ represents the same quandle class as q. However, the path $\overline{m_Q}q$ starts and ends at the basepoint, and hence represents some element of the fundamental group. Thus (abusing the notational difference between paths and their equivalence classes) $q = m_Q\overline{m_Q}q = m_Q g$.$\square$

This shows in addition that the group action on Q is transitive.

Lemma 2 *$m_Q g = m_q h$ iff $h = ag$ for some $a \in P$.*

Proof: If $h = ag$ then it is straightforward to see that $m_Q g = m_q h$. On the other hand if $m_Q g = m_q h$, then there is a map of the disk into M as shown in Fig. 2. But this disk shows that the path which follows $\overline{h}\overline{m_Q}am_Q g$ is contractible. However, $\overline{m_Q}am_Q$ is in the peripheral group, and we have our result.$\square$

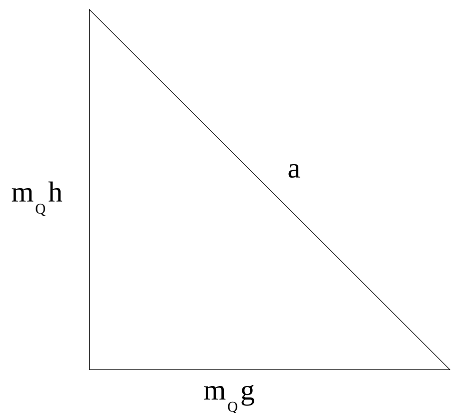

Figure 2: There is a homotopy of $m_Q g$ to $m_Q h$ with the endpoint of the path tracing out a path a on ∂N.

A similar argument shows that the stabilizer of $m_Q g$ is $g^{-1}Pg$. For elements in that conjugate peripheral group are easily seen to stabilize $m_Q g$. In particular if $p \in P$, then $m_Q g(g^{-1}pg) = m_Q pg = m_Q g$. On the other hand if some some element of the fundamental group stabilizes $m_Q g$ we may build another homotopy disk as in the previous lemma. As the stabilizer does not play a role in our following argument we leave the details of the construction as an exercise.

Let us now turn our attention to the quandle G which is built from the fundamental group and the peripheral information. Our goal is to establish the following result:

Theorem 1 *$Q \cong G$ as quandles; hence the quandle can be recovered from the fundamental group and peripheral information.*

We begin by noting that the underlying set of G is just the set of right cosets of $\pi_1(M-K)$. Therefore $\pi_1(M-K)$ naturally acts on G, and this action is obviously transitive. Indeed, every element of G is of the form Pg for some $g \in \pi_1(K)$. Furthermore, $Pg = Ph$ iff $h = ag$ for some $a \in P$.

We now define a quandle morphism $Q \to G$ sending $m_Q g \mapsto Pg$. This is well defined and invertible by our previous lemmas, so we need only show that it is a quandle map as well. Now

$$(m_Q g)^{m}_{Q} h = m_Q g\overline{h}\overline{m_Q} m m_Q h = m_Q g\overline{h} m h. \tag{4}$$

This is mapped to

$$Pg\overline{h}mh, \tag{5}$$

which is precisely Pg^{Ph}. That the inverse map is also a morphism of quandles follows similarly. Therefore this is an invertible quandle morphism, and hence an isomorphism of quandles, establishing Thm. 1.$\square$

4 Remarks

Joyce has also proved a theorem for classical knots stating that the triple $(\pi_1(M-K), P, m)$ can be reconstructed from $Q(M-K)$. This is because $\pi_1(M-K)$ is isomorphic to $Adconj(Q)$, the free group on $Q(M-K)$ quotiented by taking all the relations on $Q(M-K)$ and treating the quandle operation as conjugation in the group. We write an element $q \in Q$ as $\hat{q}$ when we wish to designate them as elements of $Adconj(Q)$. Note that $Adconj(Q)$ is canonically isomorphic to $\pi_1(M-K)$ (up to a choice of basepoint), by sending $\hat{q}$ to the homotopy class of a Wirtinger generator, $[\overline{q}m_Q q]$. Then in $Adconj(Q)$ one may take m to be any of the elements of the generators defined. $Adconj(Q)$ acts on Q by $q\hat{q_0}...\hat{q_n} = (...(q^{q_0})^{q_1}...)^{q_n}$. If an element of $Adconj(Q)$ has two presentations, $\hat{q_0}...\hat{q_n}$ and $\hat{p_0}...\hat{p_m}$, this implies that those products of Wirtinger generators are homotopic rel basepoints, and this homotopy passes to a homotopy of $(...(q^{q_0})^{q_1}...)^{q_n}$ to $(...(q^{p_0})^{p_1}...)^{p_m}$. It is then straightforward to check that this action of $Adconj(Q)$ is just the geometrically defined action of $\pi_1(M-K)$ on Q. Then P will be the stabilizer of $m \in Q$ under this action, and so the triple can be recovered from the quandle.
However, this only permits us to recover the triple when all the relations on $\pi_1(M)$ can be written down in terms of conjugation. If $\pi_1(M)$ has no such presentation, then this will not be possible. Checking this condition is complicated, however, by the fact that Q will generally be infinite, and so the presentation we obtain will not be a finite presentation. Nonetheless, for higher-dimensional knots in spheres, the above argument shows:

Theorem 2 *For codimension two knots in spheres of any dimension, the quandle and the triple $(\pi_1(M-K), P, m)$ can both be calculated from the other. More generally this holds for codimension two knots in simply connected spaces.*

Thm. 1 does imply that any invariant based upon quandles for any knot should be interpretable as an invariant based upon classical knot invariants. Thus, for example, the quandle cocycle invariants[1] are determined by the triple $(\pi_1(M-K), P, m)$, even for higher-dimensional knots. This becomes even more powerful when it is recalled that the peripheral group of a sphere is cyclic, and hence is determined by m. On the other hand, since in general $P \cong (\pi_1(K)/L) \times \mathbb{Z}$ for some normal subgroup L, this indicates that for higher-dimensional knots which as manifolds have large fundamental groups, the quandle has the potential to capture more information than for those with smaller fundamental groups.
Eisermann[2] has in fact shown that the quandle cocycle invariants of classical

knots is a specialization of certain colourings of their fundamental groups. His construction makes use of the full peripheral structure of the classical knot (that is, the longitude and the meridian), and hence does not generalize immediately to higher dimensions. However, in light of our result here, we pose the question of whether a similar construction might not be possible in higher dimensions.

As a last application of our result, we give the following theorem. Recall that *virtual* knots are a combinatorial generalization of knot diagrams introduced by Kauffman[4], and welded knots are a quotient of virtual knots given by adding an additional permitted move, first explored for braids in [3]. When the diagrams are allowed to have two endpoints instead of being closed we obtain virtual (welded) *arcs*[8, 1].

Theorem 3 *The fundamental quandle of a virtual or welded knot (or arc) K determines, and is determined by the triple $(\pi_1(K), P, m)$.*

Proof: Satoh[8] has defined a map $Tube$ on welded knots and arcs (hence on virtual knots and arcs) which maps each knot or arc to a surface knot, and shown that $Tube$ preserves the quandle and fundamental group. In addition, a direct calculation shows that P and m are preserved by $Tube$[9]. But the latter determine the quandle for the surface knot, and hence also for the virtual or welded knot or arc.

To see that the quandle determines the triple $(\pi_1(K), P, m)$, it suffices to check this for surface knots, as was done in Thm. 2.$\square$

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